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LETTER TO THE EDITOR

Asymptotic Behavior of the Daubechies Filters

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Abstract —We discuss the asymptotic behavior of the phase of the Daubechies filters when the length of the filter goes to $+\infty$. We study especially the minimum-phased filters and the almost linear phased filters. © 1995 Academic Press, Inc.

The Daubechies filters $m_N(\xi)$ are defined in the following way [2]:

(i) $m_N(\xi)$ is a trigonometric polynomial of degree $2N+1$,

$$m_N(\xi) = \sum_{k=0}^{2N+1} a_{N,k} e^{-ik\xi}, \tag{1}$$

with real-valued coefficients $a_{N,k}$.

(ii) $\sqrt{2}m_N(\xi)$ and $\sqrt{2}e^{-i\xi}\bar{m}_N(\xi+\pi)$ are conjugate quadrature filters:

$$|m_N(\xi)|^2 + |m_N(\xi+\pi)|^2 = 1. \tag{2}$$

(iii) $m_N(\xi)$ satisfies at 0 and π

$$m_N(0) = 1 \tag{3}$$

$$\text{for } p \in \{0,1,\dots,N\}, \quad \frac{\partial^p}{\partial \xi^p} m_N(\pi) = 0. \tag{4}$$

Conditions (1) to (4) do not define m_N in a unique way; they determine only the modulus of m_N:

$$|m_N(\xi)|^2 = Q_N(\cos\xi) \tag{5}$$

$$Q_N(X) = \left(\frac{1+X}{2}\right)^{N+1} \sum_{k=0}^{N} \binom{N+k}{k} \left(\frac{1-X}{2}\right)^k. \tag{6}$$

The behavior of $|m_N|$ is easily described when $N \to +\infty$, since we have

$$Q_N(X) = \sum_{k=0}^{2N+1} \chi_{[1/2,1]}\left(\frac{k}{2N+1}\right) \times \binom{2N+1}{k} \left(\frac{1+X}{2}\right)^k \left(\frac{1-X}{2}\right)^{2N+1-k}. \tag{7}$$

Thus Q_n is a Bernstein polynomial [1, 5, 6] and we have

$$\lim_{N\to+\infty} Q_N(X) = 0 \text{ if } X \in [-1,0), \quad \frac{1}{2} \text{ if } X = 0, 1 \text{ if } X \in (0,1]. \tag{8}$$

The study of the phase of m_N is much more difficult. But, surprisingly enough, it can be done with great precision [3, 4]. We present here the main results.

We know that $m_N(\xi) = ((1+e^{-i\xi})/2)^{N+1}\mu_N(\xi)$ where

$$|\mu_N(\xi)|^2 = \sum_{k=0}^{N} \binom{N+k}{k} \left(\frac{1-\cos\xi}{2}\right)^k = \prod_{k=1}^{N} \frac{\cos\xi - X_{n,k}}{1 - X_{N,k}}. \tag{9}$$

The roots $X_{N,k}$ all have multiplicity 1 and may be ordered so that $X_{N,N+1-k} = \bar{X}_{N,k}$ for $1 \leq k \leq N$ and $\operatorname{Im} X_{N,k} > 0$ for $1 \leq k \leq [N/2]$; moreover, we have $\operatorname{Re} X_{N,k} > 0$ for all k. Now, if $z_{N,k}$ is defined by $\frac{1}{2}(z_{N,k} + 1/z_{N,k}) = X_{N,k}$ and $|z_{N,k}| > 1$, we have

$$\mu_N(\xi) = \prod_{k=1}^{N} \frac{e^{-i\xi} - z_{N,k}^{\epsilon_{N,k}}}{1 - z_{N,k}^{\epsilon_{N,k}}}, \quad \epsilon_{N,k} = \pm 1, \epsilon_{N,k} = \epsilon_{N,N+1-k}. \tag{10}$$

Thus we have exactly $2[(N+1)/2]$ choices for the phase of μ_N.

Notation. If $Z_1,\dots,Z_N$ are N complex numbers such that for all $k \in \{1,\dots,N\}, |Z_k| \neq 1$, we define the phase $\omega(Z_1,\dots,Z_N)(\xi)$ of the function $\prod_{k=1}^{N}(e^{-i\xi} - Z_k)/(1-Z_k)$ as the $\mathscr{C}^\infty$ real-valued function ω such that $\omega(0) = 0$ and $\prod_{k=1}^{N}(e^{-i\xi} - Z_k)/(1-Z_k) = e^{-i\omega(\xi)} \prod_{k=1}^{N} |(e^{-i\xi} - Z_k)/(1-Z_k)|$.

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This function is easily computed by the formula

$$\omega(Z_1,\dots,Z_N)(\xi) = \operatorname{Im}\left(\int_0^{\xi}\sum_{k=1}^{N}\frac{ie^{-is}}{e^{-is}-Z_k}\,ds\right). \quad (11)$$

We may now state the theorem we proved on the phase of the Daubechies filter:

THEOREM. *Let $|\mu_N(\xi)|^2$ be given by (9), let the roots $X_{N,1},\dots,X_{N,k}$ be ordered by*

(i) *for* $1 \leqslant k \leqslant [(n+1)/2]$, $\operatorname{Im} X_{N,k} \geqslant 0$ *and* $X_{N,N+1-k} = \overline{X_{N,k}}$

(ii) $|X_{N,1}| < |X_{N,2}| < \cdots < |X_{N,[(N+1)/2]}|$ *and let* $z_{N,k}$ *be defined by* $X_{N,k} = \frac{1}{2}(z_{N,k} + 1/z_{N,k})$ *and* $|z_{N,k}| > 1$.

(A) *Phase of a general Daubechies filter. For* $1 \leqslant k \leqslant N$, $z_{N,k}$ *can be approximated by* $Z_{N,k}$ *where:*

(j) *For* $1 \leqslant k \leqslant [N^{1/5}/\log N]$, $Z_{N,k} = i - \bar{\gamma}_k/\sqrt{N}$, *where* $\gamma_1,\gamma_2,\dots,\gamma_k,\dots$ *are the roots of* $\operatorname{erfc}(z) = 1-(2/\sqrt{\pi})\int_0^z e^{-s^2}\,ds$, *such that* $\operatorname{Im}\gamma_k > 0$ *and ordered by* $|\gamma_1| < |\gamma_2| < \cdots < |\gamma_k| < \cdots$

(jj) *For* $[N^{1/5}/\log N] < k \leqslant [(N+1)/2]$, $Z_{N,k} = \theta_{N,k} + \sqrt{\theta_{N,k}^2 - 1}$, *where*

$$\operatorname{Im}\theta_{N,k} > 0 \quad \textit{and}$$

$$1-\theta_{N,k}^2 = \left(1+\frac{1}{N}\operatorname{Log}\left(2\sqrt{2N\pi\sin\varphi_{N,k}}\right)\right)e^{-2i\varphi_{N,k}} \quad (12)$$

$$\varphi_{N,k} = \frac{8k-1}{8N+6}\pi. \quad (13)$$

(jjj) *For* $\left[\frac{N+1}{2}\right] < k \leqslant N$, $Z_{N,k} = \bar{Z}_{N,N+1-k}$.

Then, whatever the choice of $\epsilon_{N,k} = \pm 1$ *(with* $\epsilon_{N,k} = \epsilon_{N,N+1-k}$*), the phase of* $\mu_N(\xi) = \prod_{k=1}^{N}(e^{-i\xi} - z_{N,k}^{\epsilon_{N,k}})/(1-z_{N,k}^{\epsilon_{N,k}})$ *satisfies*

$$\textit{for all } \xi \in \mathbb{R},\quad |\omega(z_{N,1}^{\epsilon_{N,1}},\dots,z_{N,N}^{\epsilon_{N,N}})(\xi) - \omega(Z_{N,1}^{\epsilon_{N,1}},\dots,Z_{N,N}^{\epsilon_{N,N}}(\xi)| \leqslant C_0\frac{(\operatorname{Log} N)^2}{N^{1/5}}, \quad (14)$$

where C_0 *does not depend on either* $N \geqslant 2$, *nor on* ξ *nor on the* $\epsilon_{N,k}$*'s. Thus, we may compute the phase with an* $o(1)$ *error.*

(B) *Minimum-phased filters. If we choose the roots outside of the unit disk (for all* k, $\epsilon_{N,k} = 1$*), we then have*

$$\textit{for all } \xi \in \mathbb{R},\quad |\omega(z_{N,1},\dots,z_{N,N})(\xi) - N\omega(\xi)| \leqslant C_0\sqrt{N}, \quad (15)$$

where C_0 *does not depend on* ξ *nor on* N *and where*

$$\omega(\xi) = \frac{1}{2\pi}(li_2(-\sin\xi) - Li_2(\sin\xi)) = -\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(\sin\xi)^{2k+1}}{(2k+1)^2}. \quad (16)$$

Thus, we have an asymptotic equivalent to the phase of the minimum-phased Daubechies filter.

(C) *Almost linear-phased filters. Let* $N \in 4\mathbb{N}(N = 4q)$ *and alternate the roots outside and inside the unit disk in the following manner: for* $1 \leqslant p \leqslant q$, $\epsilon_{N,4p-3} = 1$, $\epsilon_{N,4p-2} = -1$, $\epsilon_{N,4p-1} = -1$, $\epsilon_{N,4p} = 1$. *(We have in that case* $\epsilon_{N,N+1-k} = \epsilon_{N,k}$.*) Then the phase of* μ_N *satisfies*

$$\textit{for all } \xi \in \mathbb{R},\quad |\omega(z_{N,1}^{\epsilon_{N,1}},\dots,z_{N,N}^{\epsilon_{N,N}})(\xi) - \frac{1}{2}N\xi| \leqslant C_0, \quad (17)$$

where C_0 *does not depend on* ξ *or* N.

Thus we have a linear phase up to an $O(1)$ *error.*

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